

6.635 Solution to Problem Set 4

Solution P4.1

In region 0, $z > 0$

$$E_{0z} = \int_{-\infty}^{\infty} dk_{\rho} (1 - R_{0t}^{TM}) \left(\frac{iIlk_{\rho}^2}{8\pi\omega\epsilon_0} \right) e^{ik_{0z}z} H_1^{(1)}(k_{\rho}\rho) \cos \phi$$

$$H_{0z} = \int_{-\infty}^{\infty} dk_{\rho} (1 + R_{0t}^{TE}) \left(\frac{iIlk_{\rho}^2}{8\pi k_{0z}} \right) e^{ik_{0z}z} H_1^{(1)}(k_{\rho}\rho) \sin \phi$$

In region t, $z < 0$

$$E_{tz} = \int_{-\infty}^{\infty} dk_{\rho} (1 - R_{t0}^{TM}) \left(-\frac{iIlk_{\rho}^2}{8\pi\omega\epsilon_t} \right) e^{-ik_{tz}z} H_1^{(1)}(k_{\rho}\rho) \cos \phi$$

$$H_{tz} = \int_{-\infty}^{\infty} dk_{\rho} (1 + R_{t0}^{TE}) \left(\frac{iIlk_{\rho}^2}{8\pi k_{tz}} \right) e^{-ik_{tz}z} H_1^{(1)}(k_{\rho}\rho) \sin \phi$$

Using Saddle Point method,

$$\begin{aligned} E_{tz} &= \sqrt{\frac{2\pi}{ik_t r}} e^{ik_t r} k_t \cos \theta (1 - R_{t0}^{TM}(\theta)) \left(-i \frac{Ilk_t^2 \sin^2 \theta}{8\pi\omega\epsilon_t} \right) \cos \phi \sqrt{\frac{2}{\pi k_t r}} \frac{1}{\sin \theta} e^{-i\frac{3\pi}{4}} \\ &= \frac{2}{r} e^{ik_t r} \cos \theta \sin \theta \cdot i \frac{Ilk_t^2}{8\pi\omega\epsilon_t} (1 - R_{t0}^{TM}(\theta)) \cos \phi \end{aligned}$$

$$\begin{aligned} H_{tz} &= \sqrt{\frac{2\pi}{ik_t r}} e^{ik_t r} k_t \cos \theta (1 + R_{t0}^{TE}(\theta)) \left(i \frac{Ilk_t^2 \sin^2 \theta}{8\pi k_t \cos \theta} \right) \sin \phi \sqrt{\frac{2}{\pi k_t r}} \frac{1}{\sin \theta} e^{-i\frac{3\pi}{4}} \\ &= -\frac{2}{r} e^{ik_t r} \sin \theta \cdot k_t \cdot i \frac{Il}{8\pi} (1 + R_{t0}^{TE}(\theta)) \sin \phi \end{aligned}$$

In the far field, $r \gg 1$

$$\langle \bar{S} \rangle \cdot \hat{r} = \frac{1}{2} \left(\frac{Il}{4\pi r} \right)^2 (k_t \omega \mu_0) [|1 - R_{t0}^{TM}(\theta)|^2 \cos^2 \theta \cos^2 \phi + |1 + R_{t0}^{TE}(\theta)|^2 \sin^2 \phi]$$

In the plane perpendicular to the dipole: $\phi = \frac{\pi}{2}$

$$\langle \bar{S} \rangle_r \propto |1 + R_{t0}^{TE}(\theta)|^2$$

When $k_0 = k_t \sin \theta$, $1 + R_{t0}^{TE}(\theta) = 2$. Then $\sin \theta = \frac{k_0}{k_t} = \sqrt{\frac{\epsilon_0}{\epsilon_t}} \Rightarrow \theta$ is critical angle.

In the plane containing the dipole: $\phi = 0$

$$\langle \bar{S} \rangle_r \propto |1 - R_{t0}^{TM}(\theta)|^2 \cos^2 \theta$$

Let $(\langle \bar{S} \rangle_r)' = 0 \Rightarrow \sin^2 \theta = \frac{2}{1 + \frac{\epsilon_t}{\epsilon_0}} \Rightarrow \theta = \sin^{-1} \sqrt{\frac{2}{1 + \frac{\epsilon_t}{\epsilon_0}}}$

Solution P4.2

The saddle point is determined by $\sin(\alpha_R - \alpha_0) = 0 \Rightarrow \alpha = \alpha_0$ is the saddle point.

$$\cos(\alpha_R - \alpha_0 + i\alpha_I) = \cos(\alpha_R - \alpha_0)\cosh\alpha_I - i\sin(\alpha_R - \alpha_0)\sinh\alpha_I$$

The convergence condition is

$$\sin(\alpha_R - \alpha_0)\sinh\alpha_I < 0$$

and the steepest descent path is determined by t

$$\cos(\alpha_R - \alpha_0)\cosh\alpha_I = 1$$

Near the saddle-point, an expansion gives

$$(\alpha_R - \alpha_0)^2 - \alpha_I^2 \approx 0$$

Because of the convergence condition, we have

$$\alpha_R - \alpha_0 = -\alpha_I$$

and the path is tilted with slope -1 .

As $\alpha_I \rightarrow \infty$, $\cosh\alpha_I \rightarrow \infty$, we must have $\cos(\alpha_R - \alpha_0) \rightarrow 0$; therefore, $\alpha_R \rightarrow \alpha_0 - \pi/2$. Whereas $\alpha_R \rightarrow \alpha_0 + \pi/2$ as $\alpha_I \rightarrow -\infty$. The steepest descent path Γ_1 is depicted in Figure 1.

Next assume $k = k_R + ik_I$ has a small imaginary part k_I . Since $\xi = k\rho = (k_R + ik_I)\rho$, the steepest descent path is now determined by

$$\cos(\alpha_R - \alpha_0)\cosh\alpha_I + \frac{k_I}{k_R} \sin(\alpha_R - \alpha_0)\sinh\alpha_I = 1$$

Near $\alpha = \alpha_0$

$$\alpha_I^2 + 2\frac{k_I}{k_R}(\alpha_R - \alpha_0)\alpha_I - (\alpha_R - \alpha_0)^2 = 0$$

and for the integral to converge

$$\alpha_I = -(k_I + |k_R|)(\alpha_R - \alpha_0)/k_R$$

Thus, the slope of the SDP is negative and steeper than the slope of Γ_1 . Note that α_I reduces to that of the lossless case when $k_I = 0$. As $\alpha_I \rightarrow \infty$, $\cosh\alpha_I \rightarrow \sinh\alpha_I \rightarrow \infty$, then

$$\cos(\alpha_R - \alpha_0) + \frac{k_R}{k_I} \sin(\alpha_R - \alpha_0) \rightarrow 0$$

$$\alpha_R \rightarrow \alpha_0 - \tan^{-1} \frac{k_R}{k_I} = \alpha_0 - (\pi/2 - \delta)$$

where $\tan\delta = k_I/k_R$, and that as $\alpha_I \rightarrow -\infty$,

$$\alpha_R \rightarrow \alpha_0 + (\pi/2 - \delta)$$

The steepest descent path Γ_2 is shown in Figure 1.