6.635 Solution to Problem Set 3

Solution P3.1

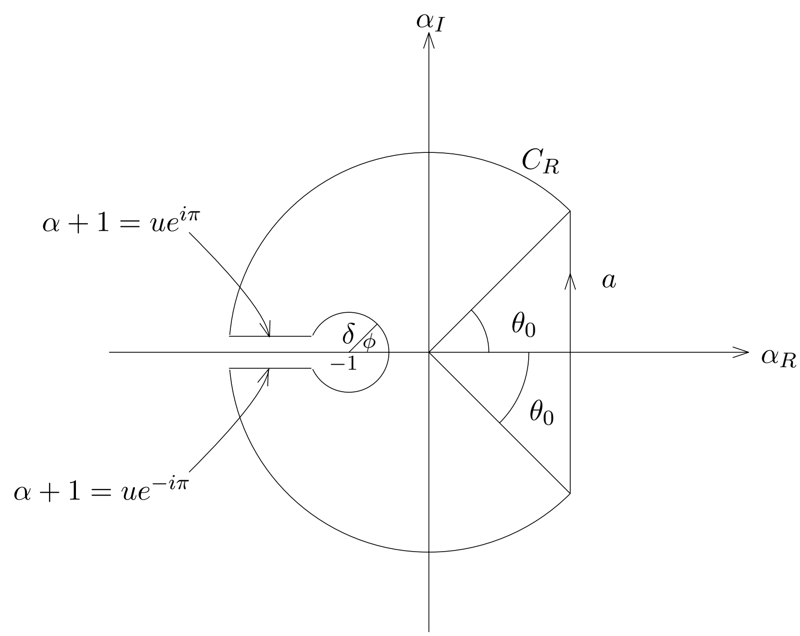

Figure 1

Using the contour shown in Figure 1 in which $\alpha = -1$ is a branch point, we can write

$$I = \frac{1}{2\pi i} \lim_{\substack{\delta \to 0 \\ R \to \infty}} \left\{ \oint_C - \int_{C_R} - \int_{C_\delta} - \int_{\Gamma_+} - \int_{\Gamma_-} \right\} \frac{e^{\alpha t}}{\sqrt{\alpha + 1}} d\alpha$$

The first integral vanishes since no poles is inside C. By Jordan's Lemma, the second integral vanishes as $R \to \infty$.

$$\lim_{\delta \to 0} \int_{C_\delta} \frac{e^{\alpha t}}{\sqrt{\alpha + 1}} dt = e^{-t} \lim_{\delta \to 0} \int_{\pi}^{-\pi} \frac{e^{\delta e^{i\phi} t}}{\sqrt{\delta e^{i\phi}}} \delta e^{i\phi} i d\phi = 0$$

On Γ_+, let $\alpha + 1 = ue^{i\pi}$, then

$$\frac{1}{\sqrt{\alpha + 1}} = \frac{1}{\sqrt{u}} e^{-i\frac{\pi}{2}} = -i \frac{1}{\sqrt{u}}$$

$$\begin{aligned} \int_{\Gamma_+} \frac{e^{\alpha t}}{\sqrt{\alpha + 1}} dt &= e^{-t} \int_{\infty}^{0} -i \frac{1}{\sqrt{u}} \, e^{-ut} du \\ &= ie^{-t} \int_0^{\infty} \frac{1}{\sqrt{u}} \, e^{-ut} du \end{aligned}$$

On Γ_-, let $\alpha + 1 = ue^{-i\pi}$, then

$$\frac{1}{\sqrt{\alpha + 1}} = \frac{1}{\sqrt{u}} \, e^{i\frac{\pi}{2}} = i \frac{1}{\sqrt{u}}$$

$$\int_{\Gamma_-} \frac{e^{\alpha t}}{\sqrt{\alpha+1}}\, dt = e^{-t} \int_0^\infty i \frac{1}{\sqrt{u}}\, e^{-ut} du$$
$$= ie^{-t} \int_0^\infty \frac{1}{\sqrt{u}}\, e^{-ut} du$$

Therefore $I = e^{-t} \int_0^\infty \frac{1}{\sqrt{u}} e^{-ut} du/\pi$. Since

$$\int_0^\infty \frac{1}{\sqrt{u}} e^{-ut} du = 2 \int_0^\infty e^{-y^2 t} dy = \sqrt{\frac{\pi}{t}}$$

thus

$$I = \sqrt{\frac{1}{\pi t}} e^{-t}$$

Solution P3.2

From the Cauchy-Riemann Condition of anlytic functions, we have

$$\frac{\partial f_R}{\partial \alpha_R} = \frac{\partial f_I}{\partial \alpha_I} \tag{P4.7.3.1}$$
$$\frac{\partial f_I}{\partial \alpha_R} = -\frac{\partial f_R}{\partial \alpha_I} \tag{P4.7.3.2}$$

Writing in terms of real variables, we further expand the contour integral to

$$\oint_C f(\alpha)\, d\alpha = \oint_C (f_R d\alpha_R - f_I d\alpha_I) + i \oint_C (f_I d\alpha_R + f_R d\alpha_I) \tag{P4.7.3.3}$$

The real part of (P4.7.3.3) is now viewed as a line integral

$$\oint_C d\overline{\ell} \cdot \overline{A}_1 = \oint_C (f_R, -f_I) \cdot (d\alpha_R, d\alpha_I)$$

By Green's theorem,

$$\oint_C d\overline{\ell} \cdot \overline{A}_1 = \iint_D d\overline{S} \cdot \nabla \times \overline{A}_1$$

we find that

$$\oint_C d\overline{\ell} \cdot \overline{A}_1 = - \iint_D d\alpha_R d\alpha_I \left(\frac{\partial f_R}{\partial \alpha_I} + \frac{\partial f_I}{\partial \alpha_R} \right) = 0$$

where the last identity follows from (P4.7.3.2). Similarly, the imaginary part of (P4.7.3.3),

$$\oint_C d\overline{\ell} \cdot \overline{A}_2 = \oint_C (f_I, f_R) \cdot (d\alpha_R, d\alpha_I)$$

By Green's theorem,

$$\oint_C d\overline{\ell} \cdot \overline{A}_2 = \iint d\alpha_R d\alpha_I \left(\frac{\partial f_R}{\partial \alpha_R} - \frac{\partial f_I}{\partial \alpha_I} \right) = 0$$

Since both the real and the imaginary parts of (P4.7.3.3) are identically zero, we have proved that

$$\oint_C f(\alpha) d\alpha = 0$$

Solution P3.3

(a) $\epsilon_R(\omega) - \epsilon_\infty = \frac{1}{\pi} PV \int_{-\infty}^{\infty} d\alpha \frac{\epsilon_I(\alpha)}{\alpha - \omega}$ $\epsilon_I(\omega) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} d\alpha \frac{\epsilon_R(\alpha) - \epsilon_\infty}{\alpha - \omega} + \frac{\sigma}{\omega}$

(b) Choose the integration path along the real α axis to be indented below $\alpha = 0$ and above $\alpha = \omega$. The residues are calculated in the same way as in part (a). However, the residue at $\alpha = 0$ should be multiplied by πi instead of $-\pi i$. In addition, there will be $2\pi i$ times the residue at $\alpha = 0$ because the contour now encloses the pole $\alpha = 0$. After cancelation, we get the same results. Similarly, if the contour is indented below $\alpha = \omega$ and above $\alpha = 0$ or indented below both $\alpha = 0$ and $\alpha = \omega$, we still get the same results.