6.635 Solution to Problem Set 2

Solution P2.1

(a) In the laboratory frame, we obtain

$$
\begin{aligned}
C_{EB} &= \overline{\overline{L}}_6^{-1} \overline{\overline{C}}_{EB}^{-1} \overline{\overline{L}}_6 \\
&= \gamma^2 \begin{bmatrix}
p'-\beta^2 q' & 0 & 0 & \ell'(1-\beta^2) & \beta(-p'+q') & 0 \\
0 & (p'-q'\beta^2) & 0 & \beta(p'-q') & \ell'(1-\beta^2) & 0 \\
0 & 0 & \dfrac{p'}{\gamma^2} & 0 & 0 & \dfrac{\ell}{\gamma^2} \\
-\ell'(1-\beta^2) & -\beta(p'-q') & 0 & (-p'\beta^2+q') & 0 & 0 \\
\beta(p'-q') & -\ell'(1-\beta^2) & 0 & 0 & (-p'\beta^2+q') & 0 \\
0 & 0 & \dfrac{-\ell}{\gamma^2} & 0 & 0 & \dfrac{q}{\gamma^2}
\end{bmatrix}
\end{aligned}
$$

(b) For a biaxial medium in its rest frame S'

$$
\begin{aligned}
\overline{\overline{\epsilon}}' &= \begin{bmatrix} \epsilon'_x & & \\ & \epsilon'_y & \\ & & \epsilon'_z \end{bmatrix} \\
\overline{\overline{C}}'_{EB} &= \begin{bmatrix} c\overline{\overline{\epsilon}}' & 0 \\ 0 & \dfrac{1}{c\mu'}\overline{\overline{I}} \end{bmatrix}.
\end{aligned}
$$

By brute force matrix multiplication, in ways exactly the same as done previously for the moving biisotropic medium, we obtain

$$
\overline{\overline{C}}_{EB} = \gamma^2 \cdot
$$

$$
\begin{bmatrix}
c\epsilon'_x - \frac{\beta^2}{c\mu'} & 0 & 0 & 0 & \beta(-c\epsilon'_x + \frac{1}{c\mu'}) & 0 \\
0 & c\epsilon'_y - \frac{\beta^2}{c\mu'} & 0 & \beta(c\epsilon'_y - \frac{1}{c\mu'}) & 0 & 0 \\
0 & 0 & \frac{c\epsilon'_z}{\gamma^2} & 0 & 0 & 0 \\
0 & \beta(-c\epsilon'_y + \frac{1}{c\mu'}) & 0 & -c\epsilon'_y\beta^2 + \frac{1}{c\mu'} & 0 & 0 \\
\beta(c\epsilon'_x - \frac{1}{c\mu'}) & 0 & 0 & 0 & -\beta^2 c\epsilon'_x + \frac{1}{c\mu'} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2 c\mu'}
\end{bmatrix}
$$

We notice that both moving biisotropic medium and moving biaxial medium are bianisotropic.

Solution P2.2

(a) In $\overline{E}$ $\overline{H}$ representation

$$
\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{\xi}} \cdot \overline{H} \tag{1}
$$

$$
\overline{B} = \overline{\overline{\zeta}} \cdot \overline{E} + \overline{\overline{\mu}} \cdot \overline{H} \tag{2}
$$

where

$$
\overline{\overline{\epsilon}} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \overline{\overline{\xi}} = \begin{bmatrix} 0 & \xi & 0 \\ -\xi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \overline{\overline{\zeta}} = \begin{bmatrix} 0 & -\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \overline{\overline{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix}.
$$

From (1) $-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot$ (2) we get

$$\overline{D}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{B}=(\overline{\overline{\epsilon}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})\cdot\overline{E}$$

Thus

$$\overline{E}=(\overline{\overline{\epsilon}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})^{-1}\cdot\overline{D}-(\overline{\overline{\epsilon}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})^{-1}\cdot\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{B}$$

Similarly

$$\overline{H}=-(\overline{\overline{\mu}}-\overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{-1}\cdot\overline{\overline{\xi}})^{-1}\cdot\overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{-1}\cdot\overline{D}+(\overline{\overline{\mu}}-\overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{-1}\cdot\overline{\overline{\xi}})^{-1}\cdot\overline{B}$$

Comparing with $\overline{D}$ $\overline{B}$ representation

$$\begin{aligned}\overline{E}&=\overline{\overline{\kappa}}\cdot\overline{D}+\overline{\overline{\chi}}\cdot\overline{B}\\ \overline{H}&=\overline{\overline{\gamma}}\cdot\overline{D}+\overline{\overline{\nu}}\cdot\overline{B}\end{aligned}$$

where

$$\overline{\overline{\kappa}}=\begin{bmatrix}\kappa & 0 & 0\\ 0 & \kappa & 0\\ 0 & 0 & \kappa_z\end{bmatrix},\overline{\overline{\chi}}=\overline{\overline{\gamma}}^{+}=\begin{bmatrix}0 & \chi & 0\\ -\chi & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\overline{\overline{\nu}}=\begin{bmatrix}\nu & 0 & 0\\ 0 & \nu & 0\\ 0 & 0 & \nu_z\end{bmatrix}.$$

Thus from

$$\begin{aligned}\overline{\overline{\kappa}}=(\overline{\overline{\epsilon}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})^{-1}&=\begin{bmatrix}1/(\epsilon-\xi^2/\mu) & 0 & 0\\ 0 & 1/(\epsilon-\xi^2/\mu) & 0\\ 0 & 0 & 1/\epsilon_z\end{bmatrix},\\ \overline{\overline{\chi}}=\overline{\overline{\gamma}}^{+}=-(\overline{\overline{\epsilon}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}})^{-1}\cdot\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{-1}&=\begin{bmatrix}0 & -\xi/(\epsilon\mu-\xi^2) & 0\\ \xi/(\epsilon\mu-\xi^2) & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\\ \overline{\overline{\nu}}=(\overline{\overline{\mu}}-\overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{-1}\cdot\overline{\overline{\xi}})^{-1}&=\begin{bmatrix}1/(\mu-\xi^2/\epsilon) & 0 & 0\\ 0 & 1/(\mu-\xi^2/\epsilon) & 0\\ 0 & 0 & 1/\mu_z\end{bmatrix}.\end{aligned}$$

Then

$$\begin{aligned}\kappa&=\frac{1}{\epsilon-\xi^2/\mu},\kappa_z=\frac{1}{\epsilon_z},\\ \chi&=\frac{-\xi}{\epsilon\mu-\xi^2},\\ \nu&=\frac{1}{\mu-\xi^2/\epsilon},\nu_z=\frac{1}{\mu_z}.\end{aligned}$$

For type I wave, when $k_y=0$, the dispersion relation is

$$k_x^2+\frac{\nu}{\nu_z}k_z^2-\frac{(\omega-\chi k_z)^2}{\kappa\nu_z}=0$$

Substitude ν, ν_z, χ and κ we get

$$k_x^2 + \frac{\mu_z}{\mu - \xi^2/\epsilon} k_z^2 - (\omega + \frac{\xi}{\epsilon\mu - \xi^2} k_z)^2 \mu_z (\epsilon - \frac{\xi^2}{\mu}) = 0$$

Thus

$$k_x^2 + \frac{\mu_z}{\mu}(k_z - \omega\xi)^2 = \omega^2 \mu_z \epsilon$$

Similarly, the dispersion relation for type II wave can be proved.

(b) For the critical angle we have $k_{xt} = 0$ thus

$$\frac{\nu}{\nu_z} k_z^2 - \frac{(\omega - \chi k_z)^2}{\kappa \nu_z} = 0$$

so the tangential wave vector will be

$$k_{zc1} = \frac{\omega}{\chi_t + \sqrt{\kappa_t \nu_t}}$$

or

$$k_{zc1} = \frac{\omega}{\chi_t - \sqrt{\kappa_t \nu_t}}$$

and the corresponding incident k_x will be

$$k_{xc1} = \sqrt{\frac{(\omega - \chi k_{zc1})^2}{\kappa \nu_z} - \frac{\nu k_{zc1}^2}{\nu_z}}$$

or

$$k_{xc2} = \sqrt{\frac{(\omega - \chi k_{zc2})^2}{\kappa \nu_z} - \frac{\nu k_{zc2}^2}{\nu_z}}.$$

Thus the critical angle will be

$$\theta_c = \arctan \frac{k_{zc1}}{k_{xc1}}$$

or

$$\theta_c = \arctan \frac{k_{zc2}}{k_{xc2}}.$$

For the incident angle θ in region 0 we have $k = \frac{k_z}{\sin\theta}$, substitude it into the dispersion relation $\omega = \chi k_z + ck/n$ we get

$$k_z = \frac{\omega}{\chi + c/n \sin\theta}.$$

Using $(\omega - \chi_t k_z)^2 - \kappa_t \nu_t k_z^2 \le 0$ we have

$$(\omega - \chi_t \frac{\omega}{\chi + c/(n \sin\theta)})^2 - \kappa_t \nu_t (\frac{\omega}{\chi + c/(n\sin\theta)})^2 \le 0$$

For $\kappa_t \nu_t \approx \frac{c^2}{n^2}$ we get

$$(\omega - \chi_t \frac{\omega}{\chi + c/(n\sin\theta)})^2 - \frac{c^2}{n^2}(\frac{\omega}{\chi + c/(n\sin\theta)})^2 \leq 0.$$

Thus

$$1 - \frac{\chi_t}{\chi + c/(n\sin\theta)} \leq \frac{c}{n(\chi + c/(n\sin\theta))}$$

So

$$\sin\theta \geq \frac{c/n}{c/n_t + \chi_t - \chi}$$

Substitude $\chi \approx c\beta(1 - \frac{1}{n^2})$ and $\chi_t \approx c\beta_t(1 - \frac{1}{n_t^2})$ we get

$$\sin\theta \geq \frac{n_t/n}{1 + n_t\beta_t(1 - 1/n_t^2) - n_t\beta(1 - 1/n^2)}.$$