6.635 Solution to Problem Set 1

Solution P1.1
Suppose that in S frame

$$\begin{aligned}\overline{E} &= E_0\hat{z};\\ c\overline{B} &= \sqrt{3}E_0\hat{z} + E_0\hat{x}.\end{aligned}$$

Consider a S' frame moving in $\hat{y}$ direction with velocity β. In S' frame,

$$\begin{aligned}E'_x &= \sqrt{3}\gamma\beta E_0\\ E'_z &= \gamma(E_0 - \beta E_0)\\ cB'_x &= \gamma(E_0 - \beta E_0)\\ cB'_z &= \sqrt{3}\gamma E_0.\end{aligned}$$

We want $\overline{E}$ and $\overline{B}$ fields to be parallel. Let a be the constant of proportionality, then

$$\begin{aligned}\sqrt{3}\gamma\beta E_0 &= a\gamma(E_0 - \beta E_0)\\ \gamma(E_0 - \beta E_0) &= a\sqrt{3}\gamma E_0.\end{aligned}$$

Eliminating E_0 and a from the above equation, we get

$$\begin{aligned}\beta^2 - 5\beta + 1 &= 0\\ \beta = \frac{5 - \sqrt{25-4}}{2} &= 0.21.\end{aligned}$$

Solution P1.2
Observer S measures $\overline{E} = \hat{x}E_0;\ c\overline{B} = \hat{y}cB_0$. For an observer S' moving with velocity $\beta\hat{z}$ with respect to S,

$$\begin{aligned}cB'_x &= 0\\ cB'_y &= \gamma(cB_0 - \beta E_0).\end{aligned}$$

Setting B'_y to 0 requires that $\beta = cB_0/E_0$. In this S' frame $E'_z = 0;\ E'_y = 0$.

$$E'_x = \gamma(E_0 - \beta cB_0) = E_0\sqrt{1 - \frac{c^2B_0^2}{E_0^2}}.$$

We learn that

$$|\overline{E}'|^2 - |c\overline{B}'|^2 = |\overline{E}|^2 - |c\overline{B}|^2$$

is an invariant quantity. Since $|\overline{E}| > |c\overline{B}|$ is the unprimed frame, $|\overline{E}|$ must be greater than $|\overline{B}|$ in all frames. We conclude that it is impossible to find an observer moving with velocity less than c who observes only a magnetic field.

Solution P1.3
From Lorentz transformation

$$ct' = \gamma(ct - \beta z) \tag{$P7.3.1$}$$
$$z' = \gamma(z - \beta ct) \tag{$P7.3.2$}$$
$$ct'' = \gamma(ct + \beta z) \tag{$P7.3.3$}$$
$$z'' = \gamma(z + \beta ct) \tag{$P7.3.4$}$$

The coordinates for event 1 in S is $ct(1, \beta)$, and, for event 2 is $ct(2, 0)$. Coordinates in S' and S'' are obtained by plugging the coordinates of S into equations (P7.3.1) to (P7.3.4). Table P7.3 is then reproduced.
(a) From the point of view of A, time elapsed is $2t$. According to A, B's speed is always v. From time dilation, time elapsed for B, according to A is $2t/\gamma$.
(b) In the initial period before turning around, B is in frame S'. For this part of the journey, time elapsed according to B from Table P7.3, is t/γ. For the second part of the journey, B is in S'', time coordinates difference is $S''_{II} - S'_{I} = ct\gamma(1 - \beta^2) = ct/\gamma$. Therefore total time elapsed, according to B is $2t/\gamma$.
(c) Relative speed of S'' with respect to S' is,

$$\beta_1 = \frac{u}{c} = -\frac{2v}{c\left(1 + \frac{v^2}{c^2}\right)} = \frac{2\beta}{1 + \beta^2} \tag{$P7.3.5$}$$

$$\gamma_1 = \frac{1 + \beta^2}{1 - \beta^2}. \tag{$P7.3.6$}$$

In the final period, for S', from Table (P7.3),

$$\Delta t' = 2\gamma t - \frac{t}{\gamma}$$
$$\Delta z' = -2\gamma\beta ct. \tag{$P7.3.7$}$$

Therefore, since B is in S'', Δct_B according to S' is

$$\Delta ct_B = \gamma_1(c\Delta t' - \beta_1 \Delta z'),$$

from (P7.3.7),

$$= \gamma_1\left(2c\gamma t - \frac{ct}{\gamma} + \frac{2\beta}{1 + \beta^2}(-2\gamma\beta ct)\right),$$
$$= \frac{ct}{\gamma}.$$

Therefore, total time elapsed from B, according to S', is also $2t/\gamma$.
(d)

$$\gamma = \frac{1}{0.6}$$

twin A will be (30/0.6 =) 5.0 years old.
(e) From table P7.3, for event 1,

$$t'' - t' = t\gamma(1 + \beta^2) - \frac{t}{\gamma} = 2\beta^2\gamma t.$$

(f) Events I and III are simultaneous for S since both occur at time t. As for S', $t'_I = t/\gamma$, $ct'_{III} = \gamma(ct)$. Therefore $t'_{III} > t'_I$. And for S'', $t''_I = \gamma t(1+\beta^2)$, $ct''_{III} = \gamma(ct)$. Therefore $t''_{III} < t''_I$. When B turns around at the time of event I is S' frame, he has not observed event III. After he has turned around, he is in S'' frame and since $t''_I > t''_{III}$, event II has already taken place. He will not observe event III either. He loses track of event III.

$$t'_{III} - t'_I = t\gamma\beta^2 \tag{P7.3.8}$$

$$t''_I - t''_{III} = \gamma t\beta^2. \tag{P7.3.9}$$

Therefore B loses track of a total time period of $2\gamma t\beta^2$ of what's happening at $z = 0$.

(g)

$$ct'_i = \frac{tc}{\gamma}$$

while

$$ct'_{III} = \gamma(ct).$$

The time difference between the two events, $t'_{III} - t'_I = \gamma t\beta^2$, is proportional to the distance separation between S and S' and is proportional to time t. This is characteristic of relativity theory.

Solution 1.4

Suppose there is an initial frame S' which moves with uniform speed v with respect to earth with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{P7.4.1}$$

and suppose at time $t = t_0$, corresponding to $t' = t'_0$, this inertial frame has its speed coinciding with that of B. Thus, at $t' = t'_0$,

$$U'_B = 0 \tag{P7.4.2}$$

and

$$\frac{dU'_B}{dt'} = g \tag{P7.4.3}$$

where U'_B is the velocity of B with rest to S'.

$$U_B = \frac{U'_B + v}{1 + \frac{U'_B v}{c^2}} \tag{P7.4.4}$$

where U_B is B's velocity w.r.t. earth. Since,

$$dU_B = \frac{dU'_B}{1 + \frac{U'_B v}{c^2}} - \frac{U'_B + v}{\left(1 + \frac{U'_B v}{c^2}\right)} \frac{v}{c^2} dU'_B \tag{P7.4.5}$$

and

$$\begin{aligned} dt &= \gamma \left(dt' + \frac{v}{c^2} dz'\right) \\ &= \gamma dt' \left(1 + \frac{vU'_B}{c^2}\right) \end{aligned} \tag{P7.4.6}$$

Dividing (P7.4.5) by (P7.4.6), we have,

$$\frac{dU_B}{dt} = \frac{dU'_B}{dt'} \frac{1}{\gamma^3 \left(1 + \frac{vU'_B}{c^2}\right)}. \qquad (P7.4.7)$$

In the limit $t' \to t'_0$

$$\frac{dU_b}{dt'} \to g; \qquad U'_B \to 0 \quad v \to U_B; \quad \gamma \to \frac{1}{\sqrt{1 - \frac{U_B^2}{c^2}}},$$

and (P7.4.7) becomes

$$\frac{dU_B}{dt} = g\left(1 - \frac{U_B^2}{c^2}\right)^{3/2}. \qquad (P7.4.8)$$

Equation (P7.4.8) is true in general because S' was chosen arbitrarily. (For deceleration, we replace g by $-g$.)

$$\int^{U_B} \frac{dU_B}{\left(1 - \frac{U_B^2}{c^2}\right)}^{3/2} = \int^t gdt. \qquad (P7.4.9)$$

Both sides of (P7.4.9) can easily be integrated and we find that for acceleration,

$$U_B = \frac{c(gt + k_1)}{\sqrt{c^2 + (gt + k_1)^2}} \qquad (P7.4.10)$$

and, for deceleration

$$U_B = \frac{c(-gt + k_2)}{\sqrt{c^2 + (-gt + k_2)^2}} \qquad (P7.4.11)$$

where k_1 and k_2 are arbitrary constants. Note that in equation (P7.4.11), U_B never exceeds c. As $t \to \infty, U_B \to c$. To determine k_1 and k_2, we use our initial conditions. Initially, at $t = 0, U_B = 0$, therefore $k_1 = 0$. Thus,

$$\frac{dz}{dt} = U_B = \frac{gct}{\sqrt{c^2 + (gt)^2}} \qquad (P7.4.12)$$

Since at $t = 0, z = 0$, therefore,

$$z = \frac{c^2}{g}\left\{\sqrt{1 + \left(\frac{gt}{c}\right)^2} - 1\right\}. \qquad (P7.4.13)$$

Solve (P7.4.13) for t_1 by substituting in

$$z = 2.15 \text{ light years} = 0.226 \text{ c}^2\text{m}$$
$$g = 10\text{m/sec}^2$$
$$c^2 = 9 \times 10^{16}\text{m/sec}^2.$$

We find,

$$\begin{aligned} ct_1 &= 0.31 \text{ c}^2\text{m}, \\ t_1 &= 3 \text{ years.} \end{aligned}$$

For the second half of the journey, we note that at $t = t_1$

$$\begin{aligned} U_B &= 0.95 \text{ cm/sec} \\ z &= 0.226 \text{ c}^2\text{m.} \end{aligned} \qquad (P7.4.14)$$

Using (P7.4.12) and (P7.4.14), we find that

$$z = -\frac{c}{g}\sqrt{c^2 + (-gt + 6 \cdot 2c)^2} + 0.55c^2. \qquad (P7.4.15)$$

At $z = 0.45 \text{ c}^2\text{m}$

$$\begin{aligned} t_2 &= \frac{6.2c}{g} = 0.62c \\ &= 6 \text{ years.} \end{aligned}$$

By symmetry, the return journey to earth takes 6 years. Thus, for a person on earth, total time elapsed for the entire journey is 12 years. As for the traveller, we use proper time

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt^2 - dz^2 \\ d\tau &= dt/\gamma. \end{aligned}$$

Therefore,

$$\begin{aligned} \tau &= \int_0^{0.31c} \frac{dt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}} + \int_{0.31c}^{0.62c} \frac{dt}{\sqrt{1 + \frac{(k_2 - gt)^2}{c^2}}} \\ &= 0.37c \simeq 3.6 \text{ years} \end{aligned}$$

For the traveller, total time elapsed for the entire journey is

$$\approx 7.2\text{years} \approx 7\text{years}.$$