6.632 Solution to Problem Set 5

Solution P5.1

(i)

$$\overline{\overline{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

$$\nabla \cdot \overline{\overline{I}} = \partial_x \hat{x} \cdot \hat{x}\hat{x} + \partial_y \hat{y} \cdot \hat{y}\hat{y} + \partial_z \hat{z} \cdot \hat{z}\hat{z} = \nabla$$

(ii)

$$\nabla \cdot (\nabla \times \nabla \times \overline{\overline{G}} - k^2 \overline{\overline{G}}) = \nabla \cdot (\overline{\overline{I}} \delta(\bar{r} - \bar{r}'))$$

$$-k^2 \nabla \cdot \overline{\overline{G}} = \nabla \delta(\bar{r} - \bar{r}')$$

(iii)
Making use of

$$\nabla \times \nabla \times \overline{\overline{G}} = \nabla\nabla \cdot \overline{\overline{G}} - \nabla^2 \overline{\overline{G}}$$

we change $(*)$ to

$$(\nabla^2 + k^2)\overline{\overline{G}} = -\overline{\overline{I}}\delta(\bar{r} - \bar{r}') + \nabla\nabla \cdot \overline{\overline{G}}$$

(iv)
From (ii),

$$\nabla \cdot \overline{\overline{G}} = \frac{1}{-k^2}(\nabla \delta(\bar{r} - \bar{r}'))$$

Plug in (iii),

$$(\nabla^2 + k^2)\overline{\overline{G}} = -\overline{\overline{I}}\delta(\bar{r} - \bar{r}') + \nabla(\frac{1}{-k^2}\nabla\delta(\bar{r} - \bar{r}'))$$

$$(\nabla^2 + k^2)\overline{\overline{G}} = -\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right)\delta(\bar{r} - \bar{r}')$$

Solution P5.2

(a) Finding $\overline{E}$ Field produced by $\overline{J}_1 = \hat{x}Il\delta(\overline{r})$ and $\overline{J}_2 = \hat{y}iIl\delta(\overline{r})$:

$$\begin{aligned}
\overline{E} =& i\omega\mu \left[\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right] \cdot \int\int\int dV' \frac{e^{ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r} - \overline{r}'|}(\hat{x} + i\hat{y})Il\delta(\overline{r}') \\
=& i\omega\mu Il \left[(\hat{x} + i\hat{y}) + \frac{1}{k^2}\nabla\nabla \cdot (\hat{x} + i\hat{y})\right] \frac{e^{ikr}}{4\pi r} \\
=& i\omega\mu \frac{Il}{4\pi} \left[(\hat{x} + i\hat{y}) + \frac{1}{k^2}\nabla\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right] \frac{e^{ikr}}{r}
\end{aligned}$$

In spherical coordinates the vector $(\hat{x} + i\hat{y})$ is

$$\begin{aligned}
\hat{x} + i\hat{y} =& \hat{r}\sin\theta\,(\cos\phi + i\sin\phi) + \hat{\phi}\,(i\cos\phi - \sin\phi) + \hat{\theta}\,(\cos\phi + i\sin\phi)\cos\theta \\
=& \hat{r}e^{i\phi}\sin\theta + \hat{\theta}e^{i\phi}\cos\theta + \hat{\phi}\,ie^{i\phi}
\end{aligned}$$

Some operations on (e^{ikr}/r)

$$\frac{\partial}{\partial x}\left(\frac{e^{ikr}}{r}\right) = \sin\theta\cos\phi\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}$$

$$i\frac{\partial}{\partial y}\left(\frac{e^{ikr}}{r}\right) = i\sin\theta\sin\phi\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{e^{ikr}}{r}\right) = e^{i\phi}\sin\theta\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}$$

The electric field $\overline{E}$ is

$$\begin{aligned}\overline{E}(\overline{r}) = i\omega\mu Il &\left\{\hat{r}\left[e^{i\phi}\sin\theta + e^{i\phi}\sin\theta\frac{1}{k^2}\left(-k^2 - \frac{2ik}{r} + \frac{2}{r^2}\right)\right]\right. \\ &\left.+\hat{\theta}\left[1 + \frac{i}{kr} - \frac{1}{k^2r^2}\right]e^{i\phi}\cos\theta + i\hat{\phi}\left[1 + \frac{ik}{kr} - \frac{1}{k^2r^2}\right]\right\}\frac{e^{ikr}}{4\pi r} \\ = -\eta\frac{ikIle^{ikr}}{4\pi r}e^{i\phi}&\left\{\hat{r}\left[\frac{i}{kr} + \left(\frac{i}{kr}\right)^2\right]2\sin\theta\right. \\ &\left.-\hat{\theta}\left[1 + \frac{i}{kr} + \left(\frac{i}{kr}\right)^2\right]\cos\theta - \hat{\phi}i\left[1 + \frac{i}{kr} + \left(\frac{i}{kr}\right)^2\right]\right\}\end{aligned} \tag{1}$$

(b) In the far field on the x-y plane $(\theta = \pi/2)$, equation (1) becomes

$$\overline{E}(\overline{r}) = \eta\frac{ikIle^{ikr}}{4\pi r}e^{i\phi}(\hat{\phi}i) = -\hat{\phi}\eta kIl\frac{e^{ikr}}{4\pi r}e^{i\phi}$$

On the x-y plane $\overline{k} = k_\rho\hat{\rho}$ (radial direction) and $r = \rho$

$$\overline{E}(\overline{r}, t) = \mathrm{Re}\left\{-\hat{\phi}\frac{\eta k_\rho Il}{4\pi\rho}e^{ik_\rho\rho}e^{i\phi}e^{-i\omega t}\right\} = -\hat{\phi}\frac{\eta k_\rho Il}{4\pi\rho}\cos(\omega t - \phi - k_\rho\rho)$$

Thus $\overline{E}$ is linearly polarized.

(c) In the far field on the x-y plane, $\theta = \pi/2$, $<\overline{S}>_\rho = \frac{\eta}{2}\left(\frac{kIl}{4\pi r}\right)^2$. Thus, the radiation pattern in x-y plane is a circle.

(d) For $\theta = 0°$ and in the far field, $\overline{E} = \eta\frac{ikIIe^{ikr}}{4\pi r}e^{i\phi}\left\{\hat{\theta} + i\hat{\phi}\right\}$. Thus, the electric field $\overline{E}$ is right hand circularly polarized.

(e) $<\overline{S}>_z = \hat{z}2\frac{\eta}{2}\left(\frac{kIl}{4\pi r}\right)^2$. Thus, the radiated power density in $\hat{z}$ is twice that in $\hat{x}$ direction.

Solution P5.3

(a) $\overline{J}(\overline{r}) = \hat{z}I_o e^{ik_z z}\delta(x)\delta(y) = \hat{z}I_o e^{ik_z z}\frac{\delta(\rho)}{2\pi\rho}$.
Let $\overline{E}(\overline{r}) = [\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla]\cdot\overline{g}(\rho, z)$, then $(\nabla^2 + k^2)\overline{g}(\rho, z) = -\hat{z}\frac{i\omega\mu I_o}{2\pi\rho}e^{ik_z z}\delta(\rho)$.
Let $\overline{g}(\rho, z) = \hat{z}g(\rho)i\omega\mu I_o e^{ik_z z}$, we have $[\frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}) - k_z^2 + k^2]g(\rho) = -\frac{\delta(\rho)}{2\pi\rho}$ which has the solution $g(\rho) = \frac{i}{4}H_0^{(1)}(k_\rho\rho)$, where $k_\rho^2 = k^2 - k_z^2$.
Then we can get $\overline{E}(\overline{r}) = -\frac{I_o}{4\omega\epsilon}e^{ik_z z}k_\rho[\hat{z}k_\rho H_0^{(1)}(k_\rho\rho) - \hat{\rho}ik_z H_1^{(1)}(k_\rho\rho)]$.

Correspondingly, $\overline{H}(\overline{r}) = -\hat{\phi}\frac{iI_o k_\rho}{4}H_0^{(1)'}(k_\rho\rho)e^{ik_z z}$.

For far field, $H_0^{(1)}(k_\rho\rho) \approx \sqrt{\frac{2}{\pi k_\rho \rho}}e^{i(k_\rho\rho + k_z z - \pi/4)}$, thus

$\overline{E}(\overline{r}) \approx \frac{I_o}{4\omega\epsilon}\sqrt{\frac{2k_\rho}{\pi\rho}}[\hat{\rho}k_z - \hat{z}k_\rho]e^{i(k_\rho\rho + k_z z - \pi/4)}$,

$\overline{H}(\overline{r}) \approx \hat{\phi}\frac{I_o}{4}\sqrt{\frac{2k_\rho}{\pi\rho}}e^{i(k_\rho\rho + k_z z - \pi/4)}$.

(b) $\text{Re}\{\overline{E} \times \overline{H}^*\} = \frac{I_o^2}{16\omega\epsilon}\frac{2|k_\rho|}{\pi\rho}(\hat{z}k_z + \hat{\rho}k_\rho) = \frac{I_o|k_\rho|}{8\omega\epsilon\pi\rho}(\hat{z}k_z + \hat{\rho}k_\rho)$.

If $k_z > k$, then k_ρ becomes imaginary, so Poynting power is only in $\hat{z}$ direction.

(c) For $k > k_z$, the phase front looks like a cone. For $k < k_z$, the phase front is perpendicular to $\hat{z}$.