6.632 Solution to Problem Set 4

Solution P4.1

When the optic axis is perpendicular to the plane of incidence, the dispersion relations are

$$k_x^2 + k_y^2 = \omega^2 \mu \epsilon \qquad \text{for the ordinary wave}$$

and

$$k_x^2 + k_y^2 = \omega^2 \mu \frac{\epsilon \epsilon_z}{\epsilon_z \cos^2\theta + \epsilon \sin^2\theta} = \omega^2 \mu \epsilon_z \qquad \text{for the extraordinary wave.}$$

where $\theta = \pi/2$ is used.

The critical angle for the ordinary wave is

$$\theta_{oc} = \sin^{-1}\sqrt{\epsilon_o/\epsilon}$$

The critical angle for the extraordinary wave is

$$\theta_{ec} = \sin^{-1}\sqrt{\epsilon_o/\epsilon_z}$$

Since $\epsilon_z < \epsilon$ for a negative uniaxial crystal, $\theta_{ec} > \theta_{oc}$. For the range of incidence angle θ such that $\theta_{ec} > \theta > \theta_{oc}$, only the extraordinary wave will be transmitted.

Solution P4.2

The problem can be solved by using the recursive method (Eqn.(3.4.3.1)) or propagation matrice method (Page 393). You can derive by hand or use *Mathematica* to solve it.

The problem is not very clear on whether the "wavelength" is in free space or in each layer. So both are OK in grading. Here we choose the latter. The reflectivity is $r = |R|^2 = 0$ and the transmissivity is $t = 1 - r = 1$ which implies that all the energy is transmitted.

The structure works as a filter, because the layers are $\lambda/4$ thick only at the particular frequency. The next lowest frequency that can have total transmission is three times higher. Therefore a signal with spectrum lower than $3f$ (f is the frequency at which the layers are $\lambda/4$-thick), only the component around f can have large transmission.

Another way to solve this problem is transmission line theory. From Equation (2.3.22) in the textbook

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{1 + \Gamma_L e^{-2ikz}}{1 - \Gamma_L e^{-2ikz}}$$

if $k(z_1 - z_0) = \frac{\pi}{2}$ then

$$Z(z_1) = Z_0 \frac{1 - \Gamma_L e^{-2ikz_0}}{1 + \Gamma_L e^{-2ikz_0}} = \frac{Z_0^2}{Z(z_0)}$$

if $k(z_1 - z_0) = \pi$ then

$$Z(z_1) = Z(z_0)$$

Let's define $Z_{l(l+1)}$ be the impedance at the boundary between region l and $(l+1)$. Then

$$Z_{7t} = \sqrt{\frac{\mu_0}{\epsilon_t}}$$
$$Z_{67} = \frac{\mu_0}{\epsilon_7}\sqrt{\frac{\epsilon_t}{\mu_0}}$$
$$Z_{56} = \frac{\epsilon_7}{\epsilon_6}\sqrt{\frac{\mu_0}{\epsilon_t}}$$
$$Z_{45} = \frac{\epsilon_6\mu_0}{\epsilon_5\epsilon_7}\sqrt{\frac{\epsilon_t}{\mu_0}} = \frac{\epsilon_6\mu_0}{\epsilon_7^2}\sqrt{\frac{\epsilon_t}{\mu_0}}$$
$$Z_{34} = Z_{45} = \frac{\epsilon_6\mu_0}{\epsilon_7^2}\sqrt{\frac{\epsilon_t}{\mu_0}}$$
$$Z_{23} = \frac{\epsilon_7^2}{\epsilon_3\epsilon_6}\sqrt{\frac{\mu_0}{\epsilon_t}} = \frac{\epsilon_7}{\epsilon_6}\sqrt{\frac{\mu_0}{\epsilon_t}}$$
$$Z_{12} = \frac{\epsilon_6\mu_0}{\epsilon_2\epsilon_7}\sqrt{\frac{\epsilon_t}{\mu_0}} = \frac{\mu_0}{\epsilon_7}\sqrt{\frac{\epsilon_t}{\mu_0}}$$
$$Z_{01} = \frac{\epsilon_7}{\epsilon_1}\sqrt{\frac{\mu_0}{\epsilon_t}} = \sqrt{\frac{\mu_0}{\epsilon_t}}$$

Using Equation 2.3.20, we can have the reflection coefficient at boundary between region 1 and 0.

$$\Gamma_L = \frac{Z_{01} - \sqrt{\frac{\mu_0}{\epsilon_t}}}{Z_{01} + \sqrt{\frac{\mu_0}{\epsilon_t}}} = 0$$

Solution P4.3

(a) $k_\rho = 88.9\ \text{m}^{-1}$
(b) $k_z = 44.4\,\text{m}^{-1}$, $k_\rho = 76.9\,\text{m}^{-1}$
(c) The time harmonic solution provides only a single frequency cylindrical wave solution. To properly view the Cerenkov radiation effect, it is necessary to examine the result in the time domain. The file "cerenkov.m" loops over a spectrum frequencies , summing the time-harmonic solutions to obtain the time domain solution at $t = \frac{0.5m}{0.9c}$, the moment the charge is at $z = 0.5\,\text{m}$. The plot may also be used to examine the constant phase front of the radiated fields. From the text, this angle should be $\theta = \cos^{-1}(\frac{\omega}{kv}) = \cos^{-1}(\frac{1}{n\beta}) = 38.2°$, which shows good agreemet with the plot.