MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

Problem Set No. 5 — 6.632 Electromagnetic Wave Theory — Issued: 03/06/2008 R
Spring Term 2008 — Due: 03/13/2008 R

Reading assignment: Section 4.2 , 4.3, 4.4 ; J. A. Kong, "*Electromagnetic Wave Theory*," EMW Publishing, 2005.

Problem P5.1

The inner product or dot product between two column vector $\overline{A}$ and $\overline{B}$, each represented as a $n \times 1$ matrix a and b can be expressed as

$$\overline{A} \cdot \overline{B} = a^T b$$

and the outer product between $\overline{A}$ and $\overline{B}$ can be expressed as

$$\overline{A}\,\overline{B} = ab^T$$

Let $\overline{A} = (\hat{x} + \hat{y} + \hat{z})$, so whereas

$$\overline{A} \cdot \overline{A} = 3$$

is a scaler, the outer product of the two vector:

$$\overline{A}\,\overline{A}$$

gives

$$\begin{bmatrix} \hat{x}\hat{x} & \hat{x}\hat{y} & \hat{x}\hat{z} \\ \hat{y}\hat{x} & \hat{y}\hat{y} & \hat{y}\hat{z} \\ \hat{z}\hat{x} & \hat{z}\hat{y} & \hat{z}\hat{z} \end{bmatrix}$$

Each column of the resulting dyadic can be view as a vector and operators that operate on the dyadic operate in a column-wise fashion:

$$\nabla \cdot \overline{\overline{C}} = \nabla \cdot \begin{bmatrix} C_{11}\hat{x}\hat{x} & C_{12}\hat{x}\hat{y} & C_{13}\hat{x}\hat{z} \\ C_{21}\hat{y}\hat{x} & C_{22}\hat{y}\hat{y} & C_{23}\hat{y}\hat{z} \\ C_{31}\hat{z}\hat{x} & C_{32}\hat{z}\hat{y} & C_{33}\hat{z}\hat{z} \end{bmatrix} = (\nabla \cdot \overline{C}_{,1})\hat{x} + (\nabla \cdot \overline{C}_{,2})\hat{y} + (\nabla \cdot \overline{C}_{,3})\hat{z}$$

(i) Show that $\nabla \cdot \overline{\overline{I}} = \nabla$

(ii) By taking the divergence of the equation

$$\nabla \times \nabla \times \overline{\overline{G}} - k^2 \overline{\overline{G}} = \overline{\overline{I}} \delta(\overline{r} - \overline{r}') \qquad \text{(P1)}$$

show that

$$-k^2 \nabla \cdot \overline{\overline{G}} = \nabla \delta(\overline{r} - \overline{r}')$$

(iii) Expand (P1), show that

$$(\nabla^2 + k^2)\overline{\overline{G}} = -\overline{\overline{I}}\delta(\overline{r} - \overline{r}') + \nabla\nabla \cdot \overline{\overline{G}}$$

(iv) Combine part (ii) and (iii), show that

$$(\nabla^2 + k^2)\overline{\overline{G}} = -\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right)\delta(\overline{r} - \overline{r}')$$

and hence by writting $\overline{\overline{G}} = \left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right) g$, all we need to solve is the scaler equation $(\nabla^2 + k^2)g = -\delta(\overline{r} - \overline{r}')$.

Problem P5.2

A *turnstile* antenna consists of two Hertzian dipoles positioned at right angles to each other with constant current distributions given by

$$\overline{J}_1 = \hat{x}Il\delta(\overline{r}) \qquad \text{and} \qquad \overline{J}_2 = \hat{y}iIl\delta(\overline{r})$$

respectively.

(a) Show that the electric field produced by this antenna is

$$\overline{E} = -\eta\frac{ikIle^{ikr}}{4\pi r}e^{i\phi}\left\{\hat{r}\left[\frac{i}{kr} + \left(\frac{i}{kr}\right)^2\right]2\sin\theta \right.$$
$$\left. -\hat{\theta}\left[1 + \frac{i}{kr} + \left(\frac{i}{kr}\right)^2\right]\cos\theta - \hat{\phi}\, i\left[1 + \frac{i}{kr} + \left(\frac{i}{kr}\right)^2\right]\right\}$$

(b) Find the total electric field in the far-field ($k_\rho\rho \gg 1$) in the x-y plane with $\theta = \pi/2$. Show that the real space-time dependence of the electric field is of the form $\cos(\omega t - \phi - k_\rho\rho)$. Note that

$$\hat{x} = \hat{r}\cos\phi\sin\theta - \hat{\phi}\sin\phi + \hat{\theta}\cos\phi\cos\theta$$
$$\hat{y} = \hat{r}\sin\phi\sin\theta + \hat{\phi}\cos\phi + \hat{\theta}\sin\phi\cos\theta$$

What is the polarization of the radiated wave in the x-y plane?

(c) Find the radiation power pattern in the x-y plane.

(d) Find the total radiated electric field on the z axis. What is the polarization of the radiated wave in the $\hat{z}$ direction?

(e) Calculate the power density radiated in the $+\hat{z}$ direction in the far field and compare it with the radiated power density in the $+\hat{x}$ direction.

Problem P5.3

(a) Find the far field electric and magnetic vectors due to a line current source with $I(z) = I_0 e^{ik_z z}$, placed along the z axis in free space.

(b) Evaluate the real part of the complex Poynting's vector in the far field. What happens if $k_z > k$?

(c) Determine the equi-phase surfaces (phase fronts) in the far field, both for $k - z < k$ and $k_z > k$. Is the real part of the Poynting's vector normal to the equi-phase surfaces?